DIFFERENTIABILITY OF SPEED FUNCTION AND FEEDBACK CONTROL OF LINEAR NONSTATIONARY SYSTEM

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Abstract: This talk is dealing with the controllability sets structure, differentiability of the speed function and feedback control of linear nonstationary systems.

Keywords: feedback control, speed control, controllability, non-stationary systems, linear systems.

1. INTRODUCTION

The problem of synthesis of positional speed control $u(t,x)$ (or feedback control) for dynamic system described by the equation

$$\dot{x} = v(t,x,u), \quad (t,x,u) \in \mathbb{R}^{1+n} \times U,$$

(U is a compactum in $\mathbb{R}^m$) remains one of difficult and insufficiently explored in the optimal control theory. The main reason obstructing to synthesize the positional speed control is absence of stability of the equation

$$\dot{x} = v(t,x,u(t,x)), \quad (t,x) \in \mathbb{R}^{1+n},$$

with respect to changing of the function $u(t,x)$ in sets of zero Lebeg measure in $\mathbb{R}^{1+n}$.

Apparently V.G. Boltyanskiy was the first who observes this anomalous behavior of the equation (1), but the simple example of such system (that is linear relative to phase coordinates and control function on the plane) was built in (Brunovski,1980a). Let us to give slightly modified example of P. Brunovski. Let examine the problem of end point control to the origin of coordinates with the system

$$\dot{x}_1 = -x_1 + u_1, \quad \dot{x}_2 = x_2 + u_2, \tag{1}$$

where $u \in U \doteq \{u = (u_1,u_2)\colon |u_1+u_2| \leqslant 1\}$. It's easy to make sure that the controllability set of the system (1) is the stripe

$$D = \{x = (x_1,x_2) \in \mathbb{R}^2\colon x_1 \in \mathbb{R},\ |x_2| < 1\},$$

and the optimal speed control is defined by the next equality

$$\widehat{u}(x_1,x_2) = \begin{cases} (0,0) \text{ if } & x_1 = x_2 = 0, \\ (+1,0) \text{ if } & x_1 < 0,\ x_2 = 0, \\ (-1,0) \text{ if } & 0 < x_1,\ x_2 = 0, \\ (0,-1) \text{ if } & 0 < x_2 < 1, \\ (0,+1) \text{ if } & -1 < x_2 < 0. \end{cases} \tag{2}$$

The system (1) closed with the control function (2)

$$\begin{cases} \dot{x}_1 = -x_1 + \widehat{u}_1(x_1,x_2), \\ \dot{x}_2 = x_2 + \widehat{u}_1(x_1,x_2), \end{cases} \tag{3}$$

has the next properties.

1°. Let $x(t,t_0,x^0)$ be the solution (defined by Caratheodory) of the system (3), $u(t,t_0,x^0) = \widehat{u}(x(t,t_0,x^0))$, then $u(t,t_0,x^0)$ is the optimal speed control of the system (1) for the point $(t_0,x^0) \in D$.

2°. Every non-trivial solution (defined in (Filippov, 1985)) of the system (3) started from D exponentially tends to zero when $t \to \infty$ but not achieves it for finite time.

This behavior takes place because the solutions of the system (3) defined by Caratheodory started from horizontal axis are solutions of the system $\dot{x}_1 = -x_1 - 1,\ \dot{x}_2 = x_2$ (if $x_1^0 > 0$), but the solutions defined by A.F. Filippov are solutions of the system $\dot{x}_1 = -x_1,\ \dot{x}_2 = x_2$ (if $x_1 > 0$). Therefore in this example there is no stability to perturbations of the positional speed control on zero measure sets. On figure 1 are shown velocity vectors v_C and v_F of the solutions defined by Caratheodory

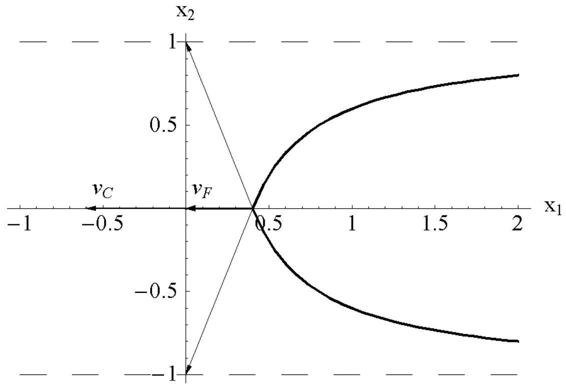

Fig 1. Velocity vectors v_C and v_F of the solutions of the system (3) defined by Caratheodory and A.F. Filippov respectively

and A.F. Filippov respectively. It should be noted that for the system (1) for every $\varepsilon > 0$ there exists ε-optimal positional speed control $u_\varepsilon(x)$, that is stable to perturbations of closed system on zero measure sets.

This talk is devoted to description of class of linear non-stationary systems with one-dimensional input that have stable positional speed control. These systems are named *subcritical systems.* For the subcritical systems are established statements about extended controllability set structure, differentiability of speed function and switching surfaces of the positional speed control.

Form the last works concerned with this themes should be noted the works of F.L. Chernousko and his learners (Chernousko, Shmatkov, 1997), (Akoolenko, Shmatkov, 1998) and the works of E.G. Albrekht and his learners (Albrekht, Ermolenko, 1997).

2. SPEED FUNCTION AND CONTROLLABILITY SET

The subject of investigations is the controllability sets structure and differentiability of the speed function $(t,x) \to \tau_n(t,x)$ of linear nonstationary system

$$\dot{x} = A(t)x + b(t)u, \tag{4}$$

$x \in \mathbb{R}^n,\ n \geqslant 2,\ |u| \leqslant 1$, where u is scalar control function, and the function $t \to (A(t), b(t))$ belongs to the class $C^r,\ r \geqslant 0$.

Some common designations are used below. $\mathbb{R}^n$ is euclidean space of dimension n with norm operator $|x| = \sqrt{x^*x}$. By Greek letters are designated row vectors and by Latin letters are column ones. Operator $*$ is transposition. $\operatorname{int} D$ is interior of set D relatively to space $\mathbb{R}^n$ and $\operatorname{cl} D$ is closure of set D in $\mathbb{R}^n$. By *support function* $\xi \to \mathrm{c}(\xi, D)$ of set D is defined the function

$$\mathrm{c}(\xi, D) = \sup\{\xi x\colon\ x \in D\}.$$

The *speed function* $(t,x) \to \tau_n(t,x)$ of the system (4) is defined by the equality

$$\tau_n(t_0, x_0) = \min_{u(\cdot)\in\mathcal{U}} \{\vartheta \geqslant 0\colon \\ x((t_0+\vartheta, t_0, x_0, u(\cdot)) = 0\},$$

where $\mathcal{U}$ is a join of measurable functions with values in $[-1,1]$ and $x((t,t_0,x_0,u(\cdot))$ is the solution of the system (4) with the control function $u = u(t)$ and starting point $x(t_0) = x_0$. *Controllability set of the system* (4) *on interval* $[t_0, t_0+\vartheta]$ is defined by the equality

$$D_\vartheta(t_0) \doteq \{x \in \mathbb{R}^n\colon\ \tau_n(t_0,x) \leqslant \vartheta\},$$

and *controllability set* of the system (4) is

$$D(t_0) \doteq \bigcup_{\vartheta \geqslant 0} D_\vartheta(t_0).$$

For the controllability sets the next equality is true (Rodionova, Tonkov, 1993):

$$D_\vartheta(t_0) = -\int_{t_0}^{t_0+\vartheta} X(t_0,t)b(t)U dt, \tag{5}$$

where $U = [-1,1],\ X(t,s)$ is the Caushi matrix of system $\dot{x} = A(t)x$ and integration is defined by Lyapunov (Ioffe, Tikhomirov, 1974). The system (4) is *differential controllable in point* t_0 if for every $\vartheta > 0$ the inclusion $0 \in \operatorname{int} D_\vartheta(t_0)$ holds. The system (4) is *differential controllable in interval* $J \subset \mathbb{R}$ if it is differential controllable in every point of J.

3. SUBCRITICALLITY

Let $\psi_1(t), \dots, \psi_n(t)$ be a fundamental sequence of solutions of the conjugate system

$$\dot{\psi} = -\psi A(t), \tag{6}$$

and let $\sigma(t)$ be the least upper bound of those $\sigma > 0$ where on half-open interval $[t, t+\sigma)$ the system of functions

$$\xi_1(t) \doteq \psi_1(t)b(t), \dots, \xi_n(t) \doteq \psi_n(t)b(t) \tag{7}$$

constitutes the Tchebyshev system (T-system). It means that every non-trivial linear combination of the functions (7) has on $[t_0, t_0+\sigma)$ not greater than $n-1$ zeroes that are geometric distinguishable.

Definition 1. The system (4) is called *subcritical* on interval J if $\sigma(t) > 0$ for every $t \in J$.

All theorems are proved for subcritical systems (4).

Lemma 1. *If the system* (4) *is subcritical on interval* J *that it is differential controllable on* J.

Example 1. Let us consider the system

$$\begin{cases} \dot{x}_1 = a_1(t)x_1 + a_2(t)x_3 + u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1 + a_3(t)x_3, \end{cases} \tag{8}$$

that describes airplane dynamics in linear approximation (Bodner, 1964). Here x_2 is the pitch angle, x_3 is the attack angle and u is the elevator angle. On figure (2) is shown parametric dependence of the function $\sigma(0)$ for the system (8) where $a_1 = 1,\ a_3 = 1$ and $-0.4 \leqslant a_2 \leqslant 0.1$.

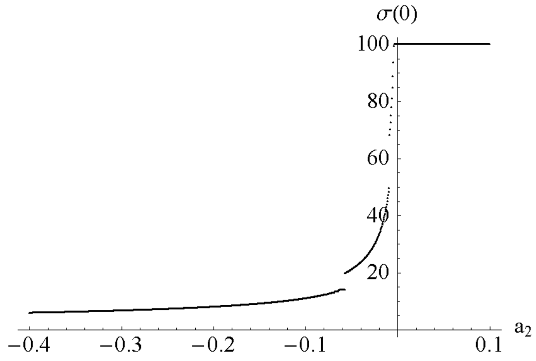

Fig 2. $\sigma(0)$ for (8), $-0.4 \leqslant a_2 \leqslant 0.1$

On figure (3) is shown the function $t \to \sigma(t)$ for the system (8) where $a_1(t) = 1,\ a_2(t) = 1,\ a_3(t) = t$ and $-2 \leqslant t \leqslant 1$.

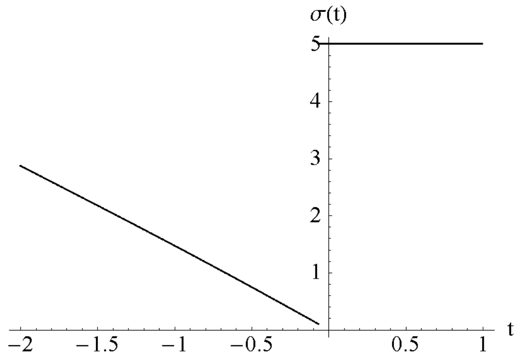

Fig 3. $\sigma(t)$ for (8), $a_1 = a_2 = 1, a_3(t) = t$

On figure (4) is shown parametric dependence of the function $\sigma(0)$ for the system (8) where $a_1(t) = \cos t$ and $-1 \leqslant a_2 \leqslant 0,\ -1 \leqslant a_3 \leqslant 1$.

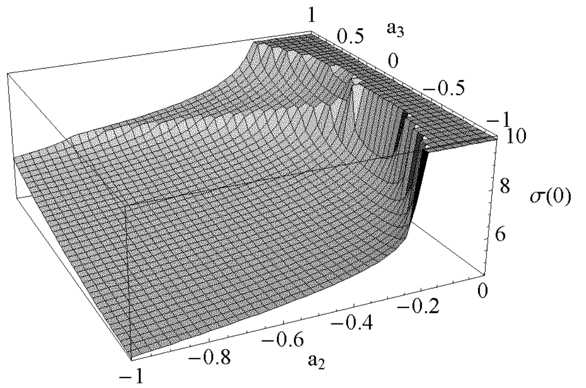

Fig 4. $\sigma(0)$ for (8), $-1 \leqslant a_2 \leqslant 0,\ -1 \leqslant a_3 \leqslant 1$

Note 1. Horizontal parts of the graphs (2)–(4) mean that on the test segments ($[0, 100]$, $[0, 5]$ and $[0, 10]$ respectively) there are no n-th zeroes of minimal linear combinations (see the section 4).

Theorem 1. *Every system of the form* (4) *that is reducible with non-singular transformation* $z(t) = L(t)x$ *(i.e.* $L(t)$ *is continuously differentiable and* $\det L(t) \neq 0,\ t \in J$*) to canonical system*

$$\dot z = F(t)z + g(t)u, \tag{9}$$

is subcritical. Here is $g(t) = \operatorname{col}(\beta_1(t), 0, \dots, 0)$,

$$F(t) = \begin{pmatrix} f_{11}(t) & f_{12}(t) & \dots & f_{1n-1}(t) & f_{1n}(t) \\ -\beta_2(t) & f_{22}(t) & \dots & f_{2n-1}(t) & f_{2n}(t) \\ 0 & -\beta_3(t) & \dots & f_{3n-1}(t) & f_{3n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\beta_n(t) & f_{nn}(t) \end{pmatrix},$$

where functions $f_{ik}(t)$ *and* $\beta_i(t)$ *are continuous,* $\beta_i(t) > 0$ *for all* $t \in J$ *and* $i = 1, \dots, n$.

Let us suppose that system (4) satisfies the following two conditions.

Condition 1. For every $i = 1, \dots, n+1$ functions defined by equalities

$$q_1(t) = b(t), \dots, q_i(t) = \dot q_{i-1}(t) - A(t)q_{i-1}(t)$$

are continuous and bounded on $\mathbb{R}$ and $\det Q(t) \neq 0$ for every $t \in \mathbb{R}$ where

$$Q(t) \doteq (q_1(t), \dots, q_n(t)).$$

Condition 2. There are numbers $\nu_1, \dots, \nu_{n-1}$ that is $\nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_{n-1}$ and for roots $\lambda_1(t), \dots, \lambda_n(t)$ of equation $\det(\lambda Q(t) - H(t)) = 0$ where

$$H(t) \doteq (q_2(t), \dots, q_{n+1}(t))$$

inequalities

$$\lambda_1(t) \leqslant \nu_1 \leqslant \lambda_2(t) \leqslant \dots \leqslant \nu_{n-1} \leqslant \lambda_n(t) \tag{10}$$

hold for every t.

Theorem 2. *If the conditions* (1) *and* (2) *are true,* $\sigma(t) = \infty$ *for all* $t \in \mathbb{R}$. *Moreover if there are constants* $\varepsilon > 0$ *and* $\delta \geqslant 0$ *that in addition to* (10) *for all sufficiently big* t *inequalities* $\delta \leqslant \lambda_1(t)$,

$$\nu_{i-1} + \varepsilon \leqslant \lambda_i(t) \leqslant \nu_i - \varepsilon, \quad i = 2, \dots, n-1,$$

are true then for all $t \in \mathbb{R}$, *controllability set* $D(t)$ *of the system* (4) *is coincide with* $\mathbb{R}^n$.

Suppose further the functions

$$A\colon \mathbb{R} \to \operatorname{End}(\mathbb{R}^n) \quad \text{and} \quad b\colon \mathbb{R} \to \mathbb{R}$$

of the system (4) be bounded on $\mathbb{R}$ and belong to class C^r (i.e. differentiable r times on $\mathbb{R}$), $r \geqslant 0$, and the system (4) is subcritical.

4. PROPERTIES AND NUMERICAL APPROXIMATION OF THE FUNCTION $\sigma(t)$

In this section for formal definition of numerical algorithm the functions

$$\xi_1(t), \dots, \xi_n(t) \tag{11}$$

will be arbitrary scalar functions bounded and continuous on segment $[t_0, t_0+\sigma]$ unless otherwise stipulated.

There are simple examples that the function $t \to \sigma(t)$ (it has non-negative finite values or $+\infty$) can be discontinuous.

Lemma 2. *Let t_0 be discontinuity point of the function $\sigma(t)$, the next inequalities are true*

$$\sigma(t_0 - 0) \leqslant \sigma(t_0) \quad \textit{and} \quad \sigma(t_0) \leqslant \sigma(t_0 + 0).$$

Let for some system of functions (11) and some point t_0 inequality $\sigma(t_0) < \infty$ is true. Without losing generality it is possible to consider in every linear combination

$$\xi(t) \doteq c_1\xi_1(t) + \cdots + c_n\xi_n(t) \qquad (12)$$

multipliers $\{c_1, \ldots, c_n\}$ satisfy the next property

$$|\operatorname{col}(c_1, \ldots, c_n)| = 1, \qquad (13)$$

because normalization of the vector

$$c \doteq \operatorname{col}(c_1, \ldots, c_n)$$

does not affect on the zeroes of the linear combination (12). Thus $c \in S^{n-1}$ and by virtue of compactness of the set S^{n-1} and linearity $\xi(t)$ by c, can be constructed convergent sequence $\{c^i\}_{i=1}^{\infty}$, to which corresponds the functional sequence $\{\xi(t; c^i)\}_{i=1}^{\infty}$ that has the next property (here $\phi_n(\xi(t))$ is n-th zero of a function $\xi(t)$):

$$\lim_{i\to\infty} \phi_n(\xi(t; c^i)) = t_0 + \sigma(t_0).$$

Appropriate limit $\widehat{\xi}(t) \doteq \lim\limits_{i\to\infty} \xi(t; c^i)$ will be call *minimal linear combination* of the functions (11).

Numerical algorithm described below is searching of linear combination (12) closest to $\widehat{\xi}(t)$.

Of course, behavior of linear combinations $\xi(t)$ of the functions (11) must be investigated on whole semi-axis $[t_0, +\infty)$, but it is impossible by virtue of limited computer resources. Therefore all computations are provided on *test segment* $[t_0, t_0 + T]$ where T is some fixed parameter of the algorithm. If there are no linear combinations of the functions (11) with n zeroes on the segment $[t_0, t_0 + T]$ then assumed $\sigma(t_0) \geqslant T$.

1°. Let $t_0, t_1, \ldots, t_{N-1}$ be the partitioning of the test segment by $N-1$ parts where N is fixed parameter of the algorithm. All values of the functions (11) and their linear combinations are computing in these N points.

According to (13) $c \in S^{n-1}$ but it is sufficient to choose multiplier sets $\{c_1, \ldots, c_n\}$ so that $c \in S_+^{n-1}$ where S_+^{n-1} can be any hemisphere of the S^{n-1}, because linear combinations (12) with $c \in S_-^{n-1}$ differ by sign only.

2°. Let $r_1, \ldots, r_n$ be an arbitrary normalized basis in $\mathbb{R}^n$. Let

$$\begin{aligned} s_2 &= r_1 \sin\theta_1 + r_2 \cos\theta_1 \\ s_3 &= s_2 \sin\theta_2 + r_3 \cos\theta_2 \\ &\ldots \\ s_n &= s_{n-1} \sin\theta_{n-1} + r_n \cos\theta_{n-1} \\ c &= \frac{s_n}{|s_n|}, \end{aligned}$$

where $0 \leqslant \theta_i < \pi$, $i = 1, \ldots, n-1$. Angles $\theta_1, \ldots, \theta_{n-1}$ are vector c coordinates in spherical coordinate system on S_+^{n-1}. Orthogonalization of the basis $r_1, \ldots, r_n$ is not required because it is simpler to normalize the vector s_n. Let M be one more fixed parameter of the algorithm. Let us separate the segment $[0, \pi]$ on $M+1$ parts and make computations of the $\xi(t)$ for every $\theta_i = \frac{2k_i\pi}{M+1}$, $k_i = 0, \ldots, M-1$, $i = 1, \ldots, n-1$. In this way on the hemisphere S_+^{n-1} there are M^{n-1} distinguishable points c, and for all of these points linear combination $\xi(t) = c_1\xi_1(t) + \cdots + c_n\xi_n(t)$ must be computed in every point of the test segment partitioning. From produced functions $\xi(t)$ let us point the one that has n-th zero closest to t_0 (corresponding vector c will be designated $\bar{c}$). If not exist let us assume $\sigma(t_0) \geqslant T$ and stop this process.

Note 2. On the second stage of the algorithm computation of the function $\xi(t)$ is required $N \cdot M^{n-1}$ times. It is clear for big n described process will take a very long time but in research purposes this algorithm is applicable.

Before to continue of the algorithm description let us examine two examples of the computed minimal linear combinations.

Example 2. On figure 5 is shown the minimal linear combination $t \to \xi(t)$ of the functions $\xi_1(t) = 1$, $\xi_2(t) = t^2$ computed with parameters $t_0 = -1$, $T = 1.3$, $N = 500$, $M = 1000$.

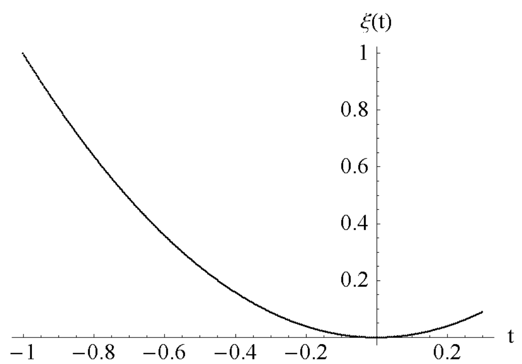

Fig 5. $\xi_1(t) = 1$, $\xi_2(t) = t^2$

Example 3. On figure 6 is shown the minimal linear combination $t \to \xi(t)$ of the functions $\xi_1(t) = t$, $\xi_2(t) = \sin(t)$, $\xi_3(t) = \cos(t)$ computed with parameters $t_0 = -1$, $T = 1.8$, $N = 500$, $M = 1000$.

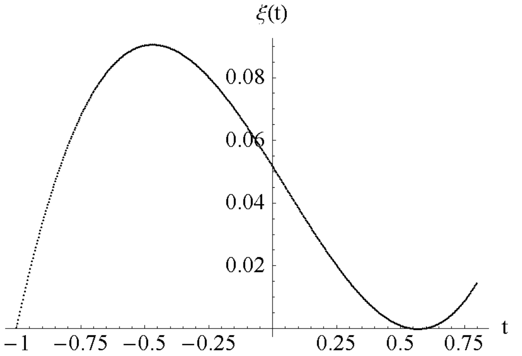

Fig 6. $\xi_1(t) = t,\ \xi_2(t) = \sin(t),\ \xi_3(t) = \cos(t)$

These examples are illustrating one important feature of minimal linear combinations. In some cases roots of computed minimal linear combination $\xi(t)$ are grouping together that in limit produces one root with multiplicity greater than 1. This event take place when the functions (11) are solutions of some differential equation or quasi-differential equation with order n. If present, it makes the next difficulty. A small changing of the multipliers $\{c_1, \dots, c_n\}$ (in other words a small shift of the vector c on the hemisphere S_+^{n-1}) produces a big changing of the n-th zero of the function $\xi(t)$. The next two stages of the algorithm are intended to avoid this effect.

3°. Let $\overline{\theta}_1, \dots, \overline{\theta}_{n-1}$ be the spherical coordinates of the vector $\overline{c}$. Let M_2 be one more parameter of the algorithm. This stage of the algorithm is repeating of the second stage with substitution of the hemisphere to the square (in spherical coordinates)

$$\left[\overline{\theta}_1 - \frac{\pi}{M-1}, \overline{\theta}_1 + \frac{\pi}{M-1}\right] \times \cdots \times \left[\overline{\theta}_{n-1} - \frac{\pi}{M-1}, \overline{\theta}_{n-1} + \frac{\pi}{M-1}\right]$$

and with substitution of the parameter M to the parameter M_2. By this way constructed more close to the minimal linear combination $\xi(t)$ with precision that can be achieve on the second stage with meaning of the parameter M equal to $M \cdot M_2$. As a matter of fact is produced $N \cdot (M^{n-1} + M_2^{n-1})$ computations of the function $\xi(t)$, that is significantly less than $N \cdot M^{n-1} \cdot M_2^{n-1}$ for big M and M_2.

4°. If after application of the algorithm's third stage the n-th and the $(n-1)$-th zeroes of the function $\xi(t)$ become more closely and distance between them become less than $\frac{2T}{N-1}$, it is proposed in limit these roots are coincide that produces one root with multiplicity greater than 1. In this case as the result of the algoritm application (or point $t_0+\sigma(t_0)$) takes the arithmetic mean of these roots. In other case as the result takes the n-th zero value. It is easy to make shure that this algorithm has an error not greater than $1/N$.

Described algorithm will be call *slow* because it can be improved in such cases when the first zero of the minimal linear combination $\widehat{\xi}(t)$ located in the point t_0. This event takes place when the functions (11) are defined by (7) and the system (4) satisfies the theorem 1. Let us designate

$$\xi_0 \doteq (\xi_1(t_0), \dots, \xi_n(t_0))$$

and separate from the sphere S^{n-1} such points c' that satisfies $\xi_0 c' = 0$. The set of these points is the sphere S^{n-2}. Let us separate from this sphere the hemisphere S_+^{n-2} by arbitrary way. The *fast* algorithm is the modification of the described above with substitution of the hemisphere S_+^{n-1} to the hemisphere S_+^{n-2}. As the result of the fast algorithm application is the linear combination $\xi(t)$ that close to the minimal $\widehat{\xi}(t)$ and has the first zero close to the point t_0.

5. CONTROLLABILITY SET STRUCTURE

According to maximum principle of Pontryagin

$$\max_{u(\cdot)\in\mathcal{U}} \psi(t)b(t)u = \psi(t)b(t)u(t), \qquad (14)$$

$t_0 \leqslant t \leqslant t_0 + \vartheta,\ \vartheta \leqslant \sigma(t_0)$, for every $x_0 \in D_\vartheta(t_0)$ there exists integer $k,\ 0 \leqslant k \leqslant n-1$ and vector $\tau \in M^k(\vartheta)$ where $M^0(\vartheta) \doteq \{0\}$,

$$M^k(\vartheta) \doteq \{\tau = (\tau_{n-k}, \dots, \tau_{n-1}) \in \mathbb{R}^k : 0 < \tau_{n-k} < \cdots < \tau_{n-1} < \vartheta\},$$

$k = 1, \dots, n-1$, such that control function transferring $x_0 = x(t_0)$ to the origin of coordinates by a minimal time has the values $+1$ and -1 with switching at points $t_0 + \tau_i,\ i = n-k, \dots, n-1$. These points corresponds to the zeroes of the function

$$\xi(t) \doteq \psi(t)b(t),$$

where $\psi(t)$ is some non-trivial solution of the system (6) and according to $\vartheta < \sigma(t_0)$ the amount of these points is not greater than $n-1$. Later on the sets $M^k(\vartheta)$ are interprets as smooth manifolds with dimensions k imbedded in $\mathbb{R}^k$.

For every $k = 0, \dots, n-1$ let us construct sets $N_+^k(t_0, \vartheta)$ and $N_-^k(t_0, \vartheta)$ as follows way. $N_+^k(t_0, \vartheta)$ is the set of points $x_0 \in D_\vartheta(t_0)$ such that is for every one there exists a point $\tau(t_0, x_0) \in M^k(\vartheta)$ such that optimal control function $u(t, x_0),\ t_0 \leqslant t \leqslant t_0 + \vartheta$ transfers point $x(t_0) = x_0$ to $x(t_0+\vartheta) = 0$ and switches at time moments $t = t_0 + \tau_i(t_0, x_0)$ only (before the first switching $u(t, x_0) = +1$). Note that the set $N_+^0(t_0, \vartheta)$ contents only the one point that can be found from the next equality

$$N_+^0(t_0, \vartheta) = \left\{ -\int\limits_{t_0}^{t_0+\vartheta} X(t_0, t)b(t)\,dt \right\}.$$

The sets $N_-^k(t_0, \vartheta)$ are identical to the sets $N_+^k(t_0, \vartheta)$ with one exception: before the first switching $u(t, x_0) = -1$. The sets $N_+^k(t_0, \vartheta)$ (and analogously $N_-^k(t_0, \vartheta)$) have following properties.

Property 1. Let $\vartheta \leqslant \sigma(t_0)$. Then

$$N_+^k(t_0, \vartheta) \subset \partial D_\vartheta(t_0)$$

and to every point $x_0 \in N_+^k(t_0, \vartheta)$ corresponds such single point $\tau(t_0, x_0) \in M^k(\vartheta)$ (the join of switching time moments) that control function $u(t, x_0)$ satisfies to the maximum principle (14) is transferring $x_0 = x(t_0)$ to $x(t_0 + \vartheta) = 0$.

In accord to property (1) for every $\vartheta \leqslant \sigma(t_0)$ and any fixed $k = 0, \ldots, n-1$ is defined the function

$$f^{-1} \colon N_+^k(t_0, \vartheta) \to M^k(\vartheta)$$

that makes correspondence from the point $x \in N_+^k(t_0, \vartheta)$ to the point $\tau \in M^k(\vartheta)$. The function $f^{-1} = f_k^{-1}$ is depends on ϑ and index k that assumed below but not accented.

Property 2. The function f^{-1} is continuous and realizes homeomorphism of the sets $N_+^k(t_0, \vartheta)$ and $M^k(\vartheta)$. The inverse function $f : M^k(\vartheta) \to N_+^k(t_0, \vartheta)$ is defined by equality

$$f(\tau) = \sum_{i=n-k-1}^{n-1} (-1)^{i-n+k} \int\limits_{t_0+\tau_i}^{t_0+\tau_{i+1}} X(t_0, t) b(t)\, dt,$$

where $\tau_{n-k-1} = 0,\ \tau_n = 0$.

Property 3. Let $\vartheta \leqslant \sigma(t_0)$. For every $k = 1, \ldots, n-1$ and any point $\tau = (\tau_{n-k}, \ldots, \tau_{n-1}) \in M^k(\vartheta)$ vectors

$$h(\tau_{n-k}) \doteq X(t_0, t_0 + \tau_{n-k}) b(t_0 + \tau_{n-k}),$$
$$\ldots$$
$$h(\tau_{n-1}) \doteq X(t_0, t_0 + \tau_{n-1}) b(t_0 + \tau_{n-1})$$

are linearly independent.

In accord to properties (1)–(3) for any $\vartheta \leqslant \sigma(t_0)$ and every $k = 1, \ldots, n-1$ the set $N_+^k(t_0, \vartheta)$ is the smooth manifold of class C^1 with dimension k imbedded in $\mathbb{R}^n$. Moreover the next theorem is proved.

Theorem 3. *Let the system* (4) *be subcritical on* $\mathbb{R}$. *Then for every* $\vartheta \leqslant \sigma(t_0)$ *the controllability set* $D_\vartheta(t_0)$ *is strictly convex in* $\mathbb{R}^n$ *(i.e.* $\operatorname{int} D_\vartheta(t_0) \neq \varnothing$ *and for any* $x, x_0 \in \partial D_\vartheta(t_0)$ *and any* $\lambda \in (0,1)$ *point* $\lambda x + (1-\lambda) x_0 \in \operatorname{int} D_\vartheta(t_0)$*). The border* $\partial D_\vartheta(t_0)$ *of the set* $D_\vartheta(t_0)$ *is the union of nonintersecting smooth (of class* C^{r+1}*) manifolds* $N_+^k(t_0, \vartheta)$ *and* $N_-^k(t_0, \vartheta),\ k = 0, 1, \ldots, n-1$ *and the union*

$$\left(\bigcup_{i=0}^{k-1} N_-^i(t_0, \vartheta) \right) \bigcup \left(\bigcup_{i=0}^{k-1} N_+^i(t_0, \vartheta) \right)$$

is the common border of manifolds $\operatorname{cl} N_+^k(t_0, \vartheta)$ *and* $\operatorname{cl} N_-^k(t_0, \vartheta)$. *In addition to every point* $x \in N_+^k(t_0, \vartheta)$ *corresponds the single control function that transfers* $x_0 = x(t_0)$ *to* $x(t_0 + \vartheta) = 0$ *and has strictly* k *switching on* $(t_0, t_0 + \vartheta)$.

Example 4. On figures 7 and 8 are shown the set $D_\vartheta(t_0)$ and the manifold $N_+^2(t_0, \vartheta)$ of the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \tag{15}$$

$|u| \leqslant 1$, with $\vartheta = 3$, $t_0 = 0$.

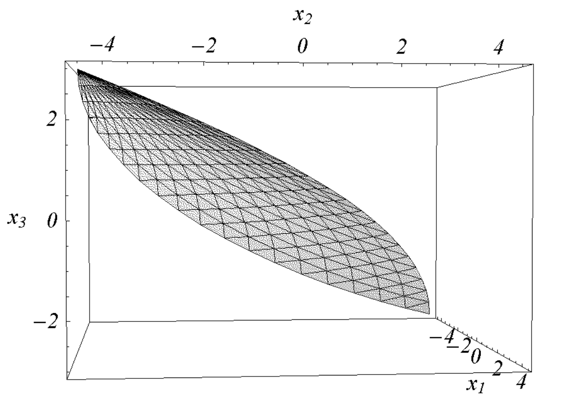

Fig 7. Set $D_\vartheta(t_0)$ for (15), $t_0 = 0,\ \vartheta = 3$

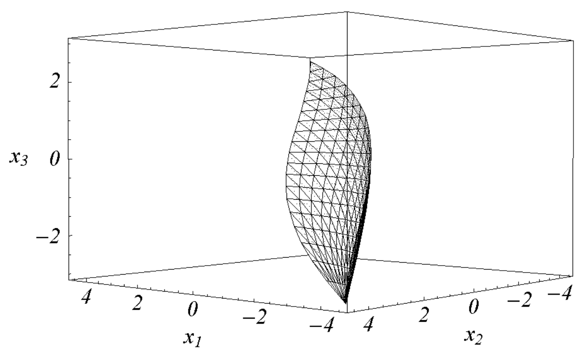

Fig 8. Manifold $N_+^2(t_0, \vartheta)$ for (15), $t_0 = 0,\ \vartheta = 3$

Example 5. On figures 9 and 10 are shown the set $D_\vartheta(t_0)$ and the manifold $N_+^2(t_0, \vartheta)$ of the system (8) with $t_0 = 0,\ \vartheta = 2\pi,\ a_1 = 1,\ a_2 = 0.1 \sin t,\ a_3 = 1 + 0.999 \sin t$.

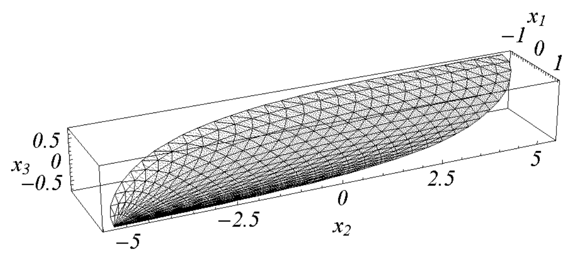

Fig 9. Set $D_\vartheta(t_0)$ for (8), $t_0 = 0,\ \vartheta = 2\pi$

6. EXTENDED CONTROLLABILITY SET STRUCTURE

Let us introduce designation $\tau_n = \vartheta$ and for every $k = 0, 1, \ldots, n$ and any $t \in \mathbb{R}$ let us define

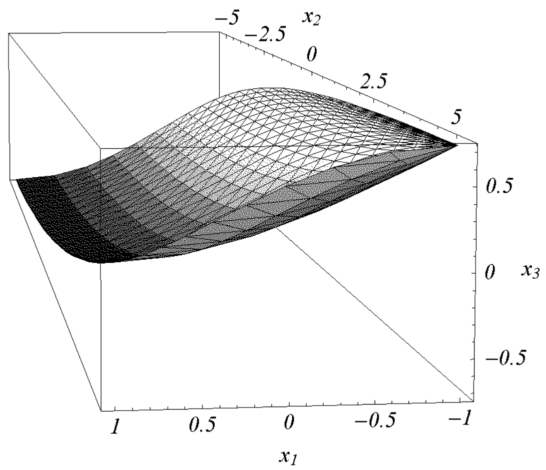

Fig 10. Manifold $N_+^2(t_0, \vartheta)$ for (8), $t_0 = 0$, $\vartheta = 2\pi$

manifolds $\mathcal{M}^k(t)$ where $\mathcal{M}^0(t) \doteq \{0\}$,

$$\mathcal{M}^k(t) \doteq \{\tau = (\tau_{n-k+1}, \dots, \tau_n)\colon \\ 0 < \tau_{n-k+1} < \dots < \tau_n < \sigma(t)\},$$

$k = 1, \dots, n$, and manifold $\mathcal{M}^{1+k} \doteq \mathbb{R} \times \mathcal{M}^k(t)$. To every point $p = (t, \tau) \in \mathcal{M}^{1+k}$ let us correspond the point $q = (t, x)$ where $x = 0$ when $k = 0$ and

$$x = x(p) = \\ -\sum_{i=n-k}^{n-1} (-1)^{i-n+k} \int\limits_{t+\tau_i}^{t+\tau_{i+1}} X(t,s)b(s)\,ds, \quad (16)$$

$\tau_{n-k} = 0$, when $k \geqslant 1$.

By the equality (16) for every k is defined the function $p \to F(p) = q$ with domain of definition $\mathcal{M}^{k+1}$ and range of values

$$\mathcal{N}_+^{1+k} \doteq F(\mathcal{M}^{1+k})$$

(lower index at $\mathcal{N}_+^1$ will be excluded below). Since $\mathcal{N}_+^{1+k} = \mathbb{R} \times \mathcal{N}_+^k(t)$ where $\mathcal{N}^0(t) = 0$ and for $k \geqslant 1$ the set $\mathcal{N}_+^k(t)$ is consist of points (16) then $\mathcal{N}_+^k(t) \subset D_{\sigma(t)}(t)$. It is proved that F is diffeomorphism of class C^{r+1} and therefore for every $k = 0, 1, \dots, n$ the set $\mathcal{N}_+^{1+k}$ is the smooth manifold of class C^{r+1}.

Theorem 4. *Let the system* (4) *is subcritical. Then extended controllability set* $\mathfrak{D} \doteq \mathbb{R} \times D_{\sigma(t)}(t)$ *can be represent as* $\mathfrak{D} = \operatorname{cl}\left(\mathfrak{N}_+^{1+n} \bigcup \mathfrak{N}_-^{1+n}\right)$ *where*

$$\mathfrak{N}_+^{1+k} = \mathcal{N}_+^{1+k} \bigcup \mathcal{N}_-^{k} \bigcup \mathcal{N}_+^{k-1} \bigcup \dots \bigcup \mathcal{N}^1,$$
$$\mathfrak{N}_-^{1+k} = \mathcal{N}_-^{1+k} \bigcup \mathcal{N}_+^{k} \bigcup \mathcal{N}_-^{k-1} \bigcup \dots \bigcup \mathcal{N}^1,$$

$k = 0, \dots, n$. *Manifolds* $\mathfrak{N}_+^{1+k}$, $\mathfrak{N}_-^{1+k}$ *are weakly invariant and for every* $k = 0, \dots, n$ *manifold* $\mathfrak{N}_+^k \bigcup \mathfrak{N}_-^k$ *is common border of the manifolds* $\operatorname{cl} \mathfrak{N}_+^{1+k}$ *and* $\operatorname{cl} \mathfrak{N}_-^{1+k}$.

Example 6. On figures 11, 12 and 13 respectively are shown the function $t \to \sigma(t)$, fragment of the union of the manifolds $\mathcal{N}_+^n \bigcup \mathcal{N}_-^n$ and fragment of the extended controllability set $\mathfrak{D}$ of the system (this system describes pendulum behavior)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -1 - 0.5\sin(2t)x_1 + u, \quad (17)$$

$|u| \leqslant 1$, with $0 \leqslant t \leqslant 6$.

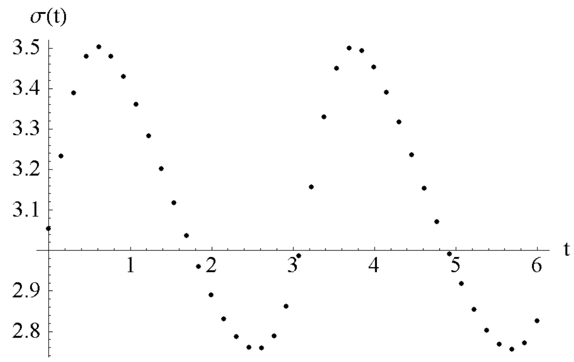

Fig 11. The function $t \to \sigma(t)$ for the system (17)

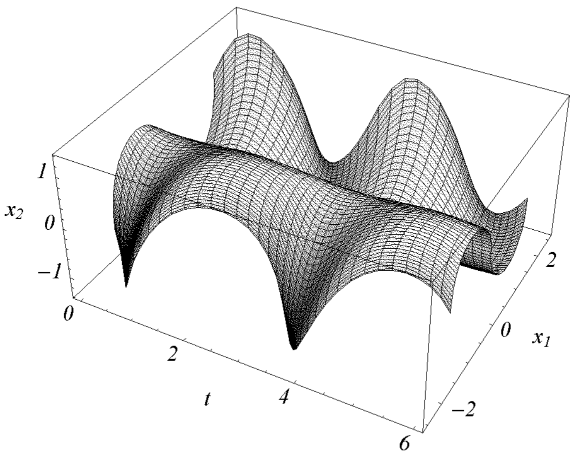

Fig 12. Fragment of the union $\mathcal{N}_+^n \bigcup \mathcal{N}_-^n$ for the system (17)

7. DIFFERENTIABILITY OF SPEED VECTOR

Let point $q_0 = (t_0, x_0) \in \mathcal{N}_+^{1+n}$ then function

$$F^{-1}\colon \mathcal{N}_+^{1+n} \to \mathcal{M}^{1+n}$$

that is reverse to F for $k = n$ give us the point

$$p_0 = (t_0, \tau_1(q_0), \dots, \tau_n(q_0)) \in \mathcal{M}^{1+n}, \\ 0 < \tau_1(q_0) < \dots < \tau_n(q_0).$$

Below it is shown that in the next problem for every $i = 1, \dots, n$ the number $\tau_i(q_0)$ is the minimal time of transferring a point $x_0 \in \mathcal{N}_+^n(t_0)$ to manifold $\mathcal{N}_{\nu(i)}^{n-i}(t_0 + \vartheta)$:

$$\vartheta(u(\cdot)) \to \min_{u(\cdot)}, \quad u(\cdot) \in \mathcal{U}, \quad (18)$$

$$\dot{x} = A(t)x + b(t)u(t), \quad t_0 \leqslant t \leqslant t_0 + \vartheta, \quad (19)$$

$$x(t_0) = x_0, \quad x(t_0 + \vartheta) \in \mathcal{N}_{\nu(i)}^{n-i}(t_0 + \vartheta), \quad (20)$$

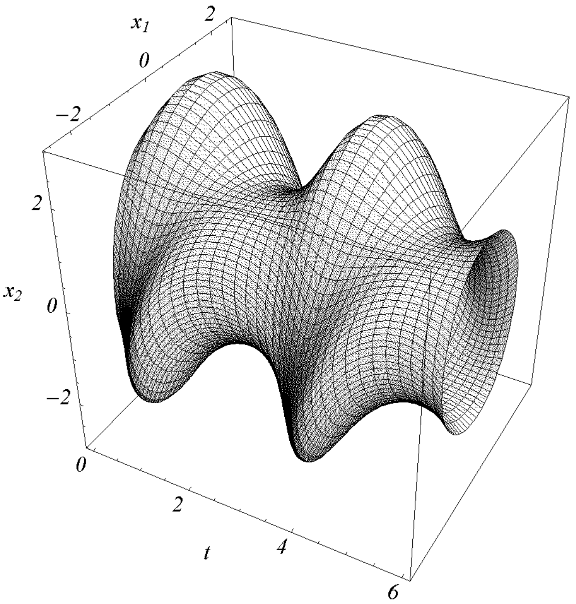

Fig 13. Fragment of the set $\mathfrak{D}$ for (17)

where $\nu(i)$ is sign "plus" if i is even and sign "minus" otherwise. In this way the vector

$$\tau(q) \doteq ((\tau_1(q), \dots, \tau_n(q))$$

is naturally to call *speed vector*. Here is $\tau_i(q) = 0$ if $q \in \mathcal{N}_{\nu(i)}^{n-i}$.

Note that the set $D_{\sigma(t)}(t)$ is centrally symmetric and therefore the vector $\tau(q)$ actually is defined on $\mathcal{N}_+^{1+n} \bigcup \mathcal{N}_-^{1+n}$.

Theorem 5. *Let the system* (4) *is subcritical. Let us designate by* $(u^0(\cdot), \vartheta^0, x^0(\cdot))$ *the optimal process of the problem* (18)–(20) *with some fixed* $i \in \{1, \dots, n\}$. *Then* $\vartheta^0 = \tau_i(q_0)$ *and on interval* $(t_0, t_0 + \tau_i(q_0))$ *the optimal control function* $u^0(t)$ *and corresponding to it optimal solution* $x^0(t)$ *of the system* (19) *are coinciding with the optimal control function and solution of the problem for transferring to the origin of coordinates (i.e. of the problem* (18)–(20) *with* $i = n$*).*

Let $\tau(q) = (\tau_1(q), \dots, \tau_n(q))$ is the speed vector of the system (4). Note that by derivative $d\tau_i(q_0)$ of function $\tau_i : \mathcal{N}_+^{1+k} \to \mathbb{R}$ in point q_0 in direction of vector $w \in T_{q_0}\mathcal{N}_+^{1+k}$ (here and below the $T_{q_0}\mathcal{N}_+^{1+k}$ is tangent space for manifold $\mathcal{N}_+^{1+k}$ in point q_0) we call the linear transformation

$$d\tau_i(q_0)\colon T_{q_0}\mathcal{N}_+^{1+k} \to \mathbb{R}$$

that is defined by the equality

$$d\tau_i(q_0)w \doteq \left.\frac{d\tau_i(q(\varepsilon))}{d\varepsilon}\right|_{\varepsilon=0}$$

where $q(\varepsilon)$ is the class of equivalence of smooth curves of kind $q : (-1,1) \to \mathcal{N}_+^{1+k}$ with the following properties: $q(0) = q_0$, $dq(\varepsilon)/d\varepsilon|_{\varepsilon=0} = w$. Similarly are defined derivatives $d^s\tau_i$, $s \geqslant 2$:

$$d^s\tau_i(q_0)\,(w_1, \dots, w_s) \doteq \left.\frac{d^s\tau_i(q(\varepsilon))}{d\varepsilon^s}\right|_{\varepsilon=0}, \qquad (21)$$

where $q : (-1,1) \to \mathcal{N}_+^{1+k}$ is the class of equivalence of smooth curves of kind

$$q(\varepsilon) = q_0 + \varepsilon w_1 + \varepsilon^2 w_2/2! + \dots + \varepsilon^s w_s/s! + o(\varepsilon^s).$$

A function $q \to \tau_i(q)$ belongs to class C^s on manifold $\mathcal{N}_+^{1+k} \bigcup \mathcal{N}_-^{1+k}$ if for any C^s-curve

$$q\colon (-1,1) \to \mathcal{N}_+^{1+k} \bigcup \mathcal{N}_-^{1+k}$$

the function $\varepsilon \to \tau_i(q(\varepsilon))$ is in the class C^s.

Theorem 6. *Let the system* (4) *is subcritical and the functions* $A\colon \mathbb{R} \to \mathrm{End}(\mathbb{R}^n)$ *and* $b\colon \mathbb{R} \to \mathbb{R}^n$ *are belong to the class* C^r. *Then for every* $k = 0, \dots, n$ *the functions*

$$\tau_i\colon \mathcal{N}_+^{1+k} \bigcup \mathcal{N}_-^{1+k} \to \mathbb{R}, \quad i = 1, \dots, n$$

are belong to the class C^{r+1}. *In particular the function* τ_i *is continuously differentiable* $r + 1$ *times on* $\mathcal{N}_+^{1+n} \bigcup \mathcal{N}_-^{1+n}$.

8. BELLMAN EQUATIONS

By virtue of the theorem 6 speed vector $\tau(q) = (\tau_1(q), \dots, \tau_n(q))$ of the system (4) where $q = (t, x)$ is differentiable along a directions tangent to corresponding manifolds. This fact permits to write Bellman equations for coordinates $\tau_i(q)$ of the speed vector in the extended controllability set $\mathfrak{D} = \mathbb{R} \times D_{\sigma(t)}(t)$.

In the first place let us to note that in all points q of the set $\mathcal{N}_+^{1+n} \bigcup \mathcal{N}_-^{1+n}$ all coordinates of the speed vector are differentiable along all directions, therefore these coordinates appear classic solutions of the equation

$$\frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial x}(A(t)x + b(t)) = -1, \qquad (22)$$

in the set $(t, x) \in \mathcal{N}_+^{1+n}$, and of the equation

$$\frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial x}(A(t)x - b(t)) = -1, \qquad (23)$$

in the set $(t, x) \in \mathcal{N}_-^{1+n}$. In addition by virtue of definition of the speed vector the function $\tau_i(t, x)$ turns into zero in all points

$$(t, x) \in \mathfrak{N}_+^{1+n-i} \bigcup \mathfrak{N}_-^{1+n-i}, \quad i = 1, \dots, n$$

(see (18)–(20)). Therefore the function $\tau_1(t, x)$ satisfies the equations (22), (23) and boundary condition

$$\tau_1(t, x) = 0 \quad \text{for all} \quad (t, x) \in \mathfrak{N}_+^n \bigcup \mathfrak{N}_-^n.$$

Further let the point $(t, x) \in \mathcal{N}_-^n \bigcup \mathcal{N}_+^n$ then the functions $\tau_2(t, x), \dots, \tau_n(t, x)$ are satisfy the equation

$$d\theta(t, x)(1, A(t)x - b(t)) = -1$$

in the set $(t,x) \in \mathcal{N}_-^n$, and the equation

$$d\theta(t,x)(1, A(t)x + b(t)) = -1$$

in the set $(t,x) \in \mathcal{N}_+^n$.

It is easy to write similar equations on other manifolds $\mathcal{N}_-^k \bigcup \mathcal{N}_+^k$, $k = n-1, \dots, 1$. And the next statement is true.

Theorem 7. *In the domain $\mathbb{R} \times D_{\sigma(t)}(t)$ the speed function $(t,x) \to \tau_n(t,x)$ of the subcritical system* (4) *is continuous solution of the problem*

$$d\theta(t,x)w(t,x) = -1, \quad \theta(t,x)|_{x=0} = 0,$$

where $d\theta(t,x)w(t,x)$ is derivative of the function $\theta(t,x)$ in the point (t,x) along the direction of the vector $w(t,x) = (1, A(t)x + u(t,x)b(t))$,

$$u(t,x) = \begin{cases} 1, & \text{if} \quad (t,x) \in \bigcup\limits_{k=1}^{n} \mathcal{N}_+^{1+k}, \\ -1, & \text{if} \quad (t,x) \in \bigcup\limits_{k=1}^{n} \mathcal{N}_-^{1+k}. \end{cases}$$

9. POSITIONAL CONTROL OF NONSTATIONARY SYSTEM

Let function $q \to u(q)$ where $q = (t,x)$ defined on the interior of the extended controllability set $\mathfrak{D}$ has values on $U = [-1,1]$ and superpositionally measurable. By *C-solution* (solution defined by Caratheodory) of system

$$\dot{x} = A(t)x + b(t)u(t,x) \tag{24}$$

will be called any absolutely continuous function $t \to x(t)$ that satisfies for all t the equality

$$x(t) = X(t,t_0)x(t_0) + \int\limits_{t_0}^{t} X(t,s)b(s)u(s,x(s))\,ds,$$

where t_0 is arbitrary fixed time moment. The main lack of the C-solutions is strong sensitivity to changes of function $u(q)$ on sets with zero measure. This lack is not peculiar to *F-solutions* (solutions defined by A.F. Filippov in (Filippov, 1985)) that are described below. Moreover F-solutions are preferable for applied problems that can be model by differential equations with discontinuities on phase coordinates.

To define the solutions in Filippov sense let us construct multiform function

$$q \to \mathbb{F}(q) \doteq \bigcap_{\varepsilon>0} \bigcap_{\mathrm{mes}\,\mu=0} \overline{\mathrm{conv}}\; u(O_\varepsilon(q) \setminus \mu), \tag{25}$$

$q \in \mathrm{int}\,\mathfrak{D}$, where $O_\varepsilon(q)$ is ε-neighborhood of the point q; μ is any set in $\mathbb{R}^{1+n}$ with zero Lebeg measure and $\overline{\mathrm{conv}}Q$ is closure of convex hull of set Q. F-solution of the system (24) is any absolutely continuous function $t \to x(t)$ that satisfies for almost all t differential inclusion

$$\dot{x} \in A(t)x + b(t)\mathbb{F}(t,x).$$

The function $u_C\colon \mathfrak{D} \to U$ that is superpositionally measurable will be called *speed optimal positional C-control* (or optimal C-control in short) if for any point $q_0 \in \mathrm{int}\,\mathfrak{D}$ the C-solution $x(t,q_0)$ of the problem

$$\dot{x} = A(t)x + b(t)u, \quad x(t_0) = x_0 \tag{26}$$

with $u = u_C(q)$ exists on semi-axis $[t_0, \infty)$, single, equal to zero when $t = t_0 + \tau_n(q_0)$ and $x(t,q_0) \equiv 0$ for $t > t_0 + \tau_n(q_0)$.

Similarly is defined *speed optimal positional F-control* (or optimal F-control in short). In this case the function $q \to u_F(q)$ must be defined for almost all (according to Lebeg measure in $\mathbb{R}^{1+n}$) points $q \in \mathrm{int}\,\mathfrak{D}$ and provide the next property: for every $q_0 \in \mathrm{int}\,\mathfrak{D}$ corresponds a single F-solution $x(t,q_0)$ of problem (26) with control function $u = u_F(q)$ and $x(t,q_0) \equiv 0$ for $t \geqslant t_0 + \tau_n(q_0)$. Note once again that by virtue of definition of F-solutions to construct the optimal F-control there are no necessity to define $u_F(q)$ in every point of interior of the extended controllability set $\mathfrak{D}$. It is sufficient to construct $u_F(q)$ on a set of full measure.

As shown in the introduction there are examples of such abnormal behavior of linear controllable systems (Brunovski P., 1980a, 1980b): the optimal C-control exists and is unique but optimal F-control does not exist. This occurence takes place (even for linear stationary systems) in case when optimal C-control that strictly defined from maximum principle of Pontryagin is defines on surfaces of discontinuity (this surfaces has zero Lebeg measure) velocity vector that is not co-directed with velocity vector from Filippov's construction (25).

Theorem 8. *Let the system* (4) *be subcritical. Then function*

$$u_C(q) = \begin{cases} 1, & \text{if} \quad q \in \mathcal{N}_+^{1+k} \text{ for any } k \\ 0, & \text{if} \quad k = 0 \\ -1, & \text{if} \quad q \in \mathcal{N}_-^{1+k} \text{ for any } k \end{cases}$$

where $k \in \{1, \dots, n\}$, is the optimal C-control, and the function

$$u_F(q) = \begin{cases} 1, & \text{if} \quad q \in \mathcal{N}_+^{1+n} \\ -1, & \text{if} \quad q \in \mathcal{N}_-^{1+n} \end{cases} \qquad q \in \mathrm{int}\,\mathfrak{D}$$

is the optimal F-control.

10. ACKNOWLEDGEMENT

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